Identification of Nonlinear Systems: Volterra Series

Antonín Novák

Dept. of Radioelectronics, Czech Technical University, Technická 2, 166 27 Praha, Czech Republic
LAUM, UMR CNRS 6613, Université du Maine, Le Mans, France

novaka4@fel.cvut.cz

Abstract. *Traditional measurement of multimedia systems, such as linear impulse response, transfer function, are sufficient but not faultless. For these methods the pure linear system is considered. Nonlinearities, which are usually included in the most of real systems are disregarded. One of the simple methods that can describe the nonlinear system used in practice is coefficient of distortion or intermodulation distortion, but these methods cannot be used to determine nonlinearities themselves.*
This paper describe one of the methods to identify nonlinear systems called Volterra Series. A simplification for this method is proposed and an experiment with audio amplifier is shown to test this method.

Keywords

Nonlinear, Volterra Series, systems, identification.

1. Introduction

Let multimedia or audio system is a black box for which there are rules of theory of signal processing. This black box is time invariant, that means the properties of black box does not change. Signal $y(t)$ is a response to a system to the excitation $x(t)$. Any given input $x_i(t)$ produces a unique output $y_i(t)$. Not only one input $x(t)$ can produce the same output $y(t)$, but not vice versa, that means there is only one response $y(t)$ to input $x(t)$.

$$\begin{aligned} y(t) = &\int_{-\infty}^{\infty} h_1(\tau_1)x(t-\tau_1)d\tau_1 \\ &+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2(\tau_1,\tau_2)x(t-\tau_1)x(t-\tau_2)d\tau_1 d\tau_2 \\ &+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_3(\tau_1,\tau_2,\tau_3)x(t-\tau_1)x(t-\tau_2)x(t-\tau_3)d\tau_1 d\tau_2 d\tau_3 \\ &\vdots \\ &+ \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} h_n(\tau_1,...,\tau_n)x(t-\tau_1)...x(t-\tau_n)d\tau_1...d\tau_n \end{aligned} \quad (1)$$

The black box with its properties can be represented as shown in Figure 1 in which the symbol H_n is called an *Volterra operator*. This theory of Volterra series was introduced in [1].

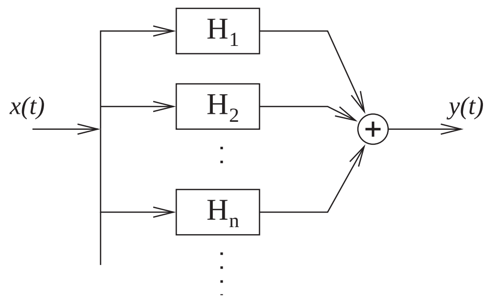


Fig. 1. Schematic representation of a Volterra series model

The relation between the output and the input can be expressed in the form given by the total sum

$$y(t) = \sum_n \mathbf{H_n}[x(t)], \quad (2)$$

in which

$$\mathbf{H_n}[x(t)] = \quad (3)$$

$$= \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} h_n(\tau_1,...,\tau_n)x(t-\tau_1)...x(t-\tau_n)d\tau_1...d\tau_n$$

represents n-dimensional convolution of the input signal $x(t)$ and n-dimensional *Volterra kernel* $h_n(\tau_1,...,\tau_n)$. Symbol $\mathbf{H_n}$ represents n-th order *Volterra operator*.

2. First-Order Volterra systems

From now on the only causal, stable and LTI (linear time invariant) *First-Order Volterra system* will be considered, for what stands

$$y(t) = \mathbf{H_1}[x(t)], \quad (4)$$

which can be expanded by using *Volterra operator* $\mathbf{H_1}$ into form

$$y(t) = \int_{-\infty}^{\infty} h_1(\tau)x(t-\tau)d\tau \tag{5}$$

This equation represents simple one-dimensional convolution, which determine pure linear system. *First-Order Volterra system* is in general linear system, in which *First-Order Volterra kernel* $h_1(t)$ is called impulse response of the system. This impulse response can be obtained by Dirac impuls excitation $\delta(t)$, from

$$h_1(t) = \mathbf{H_1}[\delta(t)] \tag{6}$$

3. Second-Order Volterra system

A linear system was considered in the last chapter. This system keeps rules of linear combination. Thats means that the response to a linear combination of input signals equals the same linear combinations of response to a input signals. The Second-Order system does not keep the rules of linear combination, but bilinear combination. The response to a linear combination of input signals equals the same bilinear combinations of response to a input signals. Let us take into consideraton a causal, stable, LTI Second-Order system, which is defined by

$$y(t) = \mathbf{H_2}[x(t)] \tag{7}$$

Operator $\mathbf{H_2}$ is called *Second-Order Volterra operator*. This operator is expressed by formula (1)

$$\mathbf{H_2}[x(t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2(\tau_1,\tau_2)x(t-\tau_1)x(t-\tau_2)d\tau_1 d\tau_2 \tag{8}$$

The function $h_2(\tau_1,\tau_2)$ is called *Second-Order Volterra kernel*. Generally, this function need not to be axis-symmetric by axis $h_2(\tau,\tau)$, but for definiteness reasons it should be better to consider this function as axis-symmetric by axis $h_2^*(\tau_1,\tau_2)$. The symmetrisation can be done by

$$h_2(\tau_1,\tau_2) = \frac{1}{2}[h_2^*(\tau_1,\tau_2) + h_2^*(\tau_2,\tau_1)] \tag{9}$$

From now on the only symmetric kernel will be considered, for which stands

$$h_2(\tau_1,\tau_2) = h_2(\tau_2,\tau_1)$$

As known from the theory of linear systems and as it is described in Eq. (6), there is a possibility to obtain the impulse response of First-Order system (linear system), as an response to a Dirac impuls.

Let us take into consideration the input signal $x(t) = \delta(t)$, which is brought into the Second-Order system. The output is given by

$$\begin{aligned} y(t) &= \mathbf{H_2}[\delta(t)] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2(\tau_1,\tau_2)\delta(t-\tau_1)\delta(t-\tau_2)d\tau_1 d\tau_2 \\ &= h_2(t,t) \end{aligned} \tag{10}$$

The response to the Dirac impuls does not determinate the Second-Order system, but represents just a slice through the axis of *Second-Order Volterra kernel* (see. Figure). Let the input signal $x(t)$ is given by sum of two signals $x_1(t)+x_2(t)$. The response to such a signal is given by

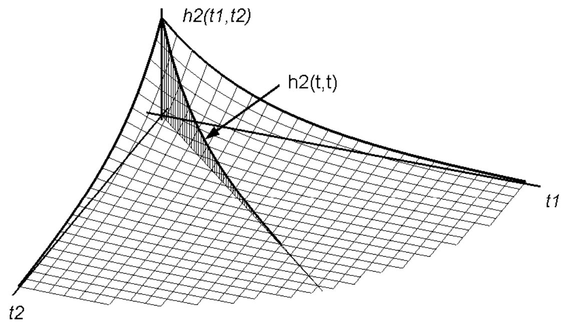


Fig. 2. Example of Second-Order Volterra Kernel

$$\begin{aligned} y(t) &= \mathbf{H_2}[x(t)] = \mathbf{H_2}[x_1(t)+x_2(t)] = \\ &= \mathbf{H_2}\{x_1(t),x_1(t)\} + 2\mathbf{H_2}\{x_1(t),x_2(t)\} + \\ &+ \mathbf{H_2}\{x_2(t),x_2(t)\} \\ &= \mathbf{H_2}[x_1(t)] + 2\mathbf{H_2}\{x_1(t),x_1(t)\} + \mathbf{H_2}[x_2(t)] \end{aligned} \tag{11}$$

in which $\mathbf{H_2}\{\cdot\}$ is *bilinear Volterra operator*, which is defined by

$$\begin{aligned} &\mathbf{H_2}\{x_1(t),x_2(t)\} = \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2(\tau_1,\tau_2)x_1(t-\tau_1)x_2(t-\tau_2)d\tau_1 d\tau_2 \end{aligned} \tag{12}$$

Thence

$$\mathbf{H_2}\{x_1(t),x_1(t)\} = \mathbf{H_2}[x_1(t)] \tag{13}$$

thus *bilinear Volterra operator* applied to two same signals is simply speaking *Second-Order Volterra operator*.

4. Higher-Order Volterra Systems

Generally the Higher-Order system can be considered, but the complexity gets higher as the order of system increases. Also the imagination of a representation of the higher order is more difficult, for the dimension is higher.
The analysis of finding all another kernels is based on finding the higher-order kernel and then recursively on finding lower-order kernels.

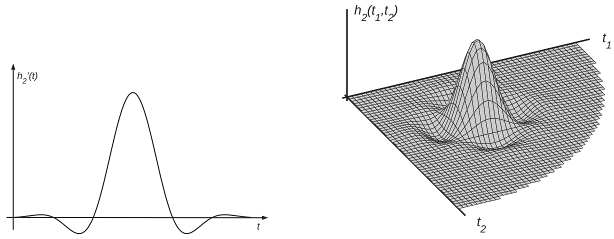
5. A simplified model

Since the n-th Volterra kernel is a function of n variables, the model which represents the system has to contain a lot of coefficients needs to determinate the system. This section describes a simplified model, which reduces the number of coefficients required for a Volterra series representation.

The first simplification replaces the n-th Volterra kernel by its symmetric representation. The Second-Order Volterra kernel will be reduced to

$$h_2(\tau_1, \tau_2) = h_2'(\tau_1) \cdot h_2'(\tau_2) \qquad (14)$$

This is demonstrated in Figure 3 showing the sub-kernel $h_2'(\tau)$ and kernel $h_2(\tau_1, \tau_2)$.



(a) sub-kernel (b) kernel

Fig. 3. The demonstration of kernel simplification

Generally for higher Volterra kernels stands that

$$h_n(\tau_1, \tau_2, ..., \tau_n) = \prod_n h_n'(\tau) \qquad (15)$$

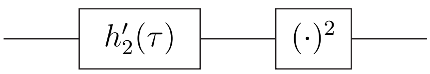


Fig. 4. Schematic representation of a simplification of a second kernel

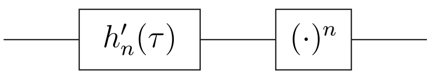


Fig. 5. Schematic representation of a simplification of a general kernel

The output signal of the Second-Order System is

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2(\tau_1, \tau_2) x(t-\tau_1) x(t-\tau_2) d\tau_1 d\tau_2 \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2'(\tau_1) h_2'(\tau_2) x(t-\tau_1) x(t-\tau_2) d\tau_1 d\tau_2 \\ &= \int_{-\infty}^{\infty} h_2'(\tau_1) x(t-\tau_1) d\tau_1 \cdot \int_{-\infty}^{\infty} h_2'(\tau_2) x(t-\tau_2) d\tau_2 \\ &= [\int_{-\infty}^{\infty} h_2'(\tau) x(t-\tau) d\tau]^2 \end{aligned} \qquad (16)$$

This equation is schematically shown in Figure 4 or in general in Figure 5.

The whole simplified Volterra model

The scheme from Figure 1 can be simplified by using the simplifications described above. The model then will not be the same as regular Volterra model, but the reduced one. The simplified Volterra model is not able to determine all the nonlinearities in the same manner as the regular Volterra model [1]. Besides, the simplified model cannot determinate another nonlinearities, but it will be shown that in some cases such as analysis of an amplifier in weakly nonlinear mode the simplified model is sufficiently precise.
The simplified model is shown in Figure 6

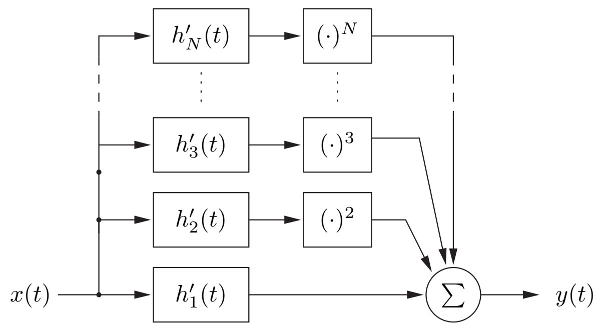


Fig. 6. The whole simplified Volterra model

The output $y(t)$ is given by

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h_1'(\tau) x(t-\tau) d\tau \\ &+ [\int_{-\infty}^{\infty} h_2'(\tau) x(t-\tau) d\tau]^2 \\ &\vdots \\ &+ [\int_{-\infty}^{\infty} h_N'(\tau) x(t-\tau) d\tau]^N \end{aligned} \qquad (17)$$

that can be rewritten into shorten form

$$y(t) = \sum_{n=1}^{N} [\int_{-\infty}^{\infty} h'_n(\tau) x(t-\tau) d\tau].^n \qquad (18)$$

6. Measuring of non-linear audio systems

Next section deals with using the theory described above. The real systems used in audio / multimedia has been measured. The method gives sufficiently precise results with respect to weak nonlinear mode. If the higher kernels are too feeble, that means if the nonlinearity is weak, it is better to use the simplest model, as the higher kernels are near the level of noise.
The method using first kernel and the set of coefficients for determinate the higher kernels has been verified on SONY SDP-300, used in the amplifier mode.

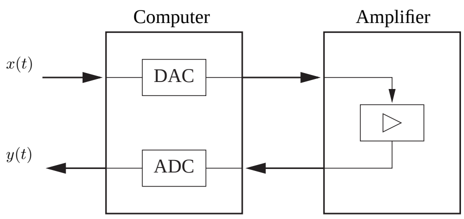


Fig. 7. Scheme of measured system

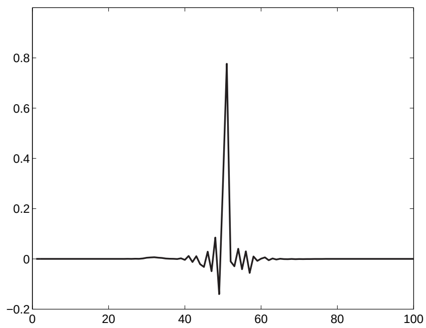


Fig. 8. The first kernel of SONY SDP-300 amplifier

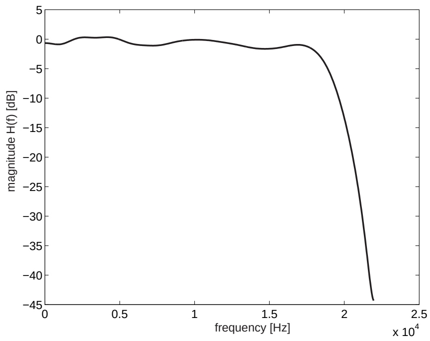


Fig. 9. Transfer function calculated from the first kernel of SONY SDP-300 amplifier

To verify the simplified Volterra model a comparison between an audio amplifier and the Volterra model has to be realized. The input signal including two harmonic signals has been put into the audio amplifier and also into the model and the output spectrum of both has been compared. Results are shown in Figures 10-13.

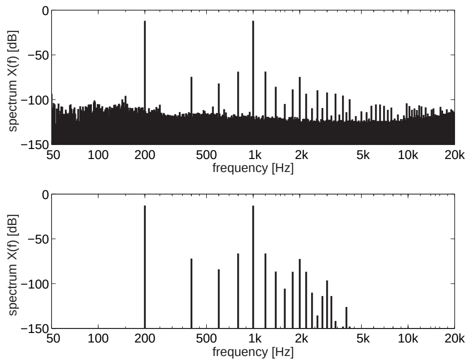


Fig. 10. Comparison of responses to $200Hz$ and $1kHz$ tones: SONY SDP-300 - above, model - below

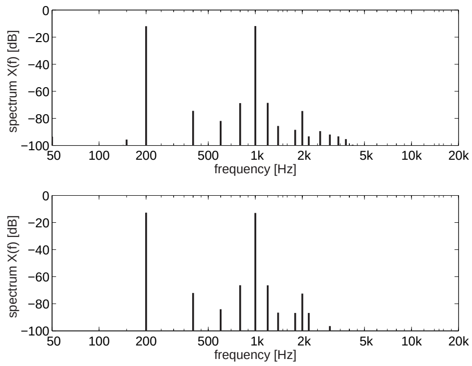


Fig. 11. Comparison of responses to $200Hz$ and $1kHz$ tones: SONY SDP-300 - above, model - below, zoomed up to -100 dB

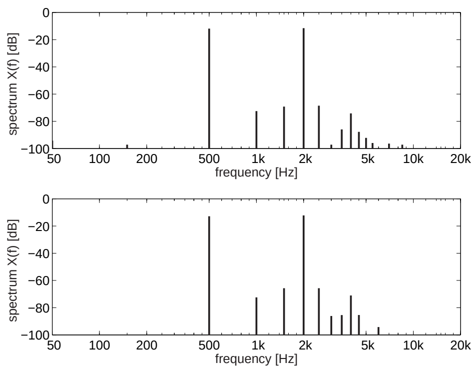


Fig. 12. Comparison of responses to $500Hz$ and $2kHz$ tones: SONY SDP-300 - above, model - below, zoomed up to -100 dB

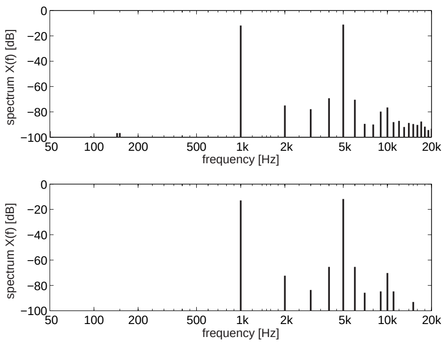


Fig. 13. Comparison of responses to $1kHz$ and $5kHz$ tones: SONY SDP-300 - above, model - below, zoomed up to -100 dB

7. Conclusion

In this paper the simplification of Volterra kernels has been presented. The simplification of kernels can be used only in systems with weak nonlinearities which can be found in multimedia systems such as amplifiers, loudspeakers etc. The results of the nonlinear model are in some cases (weak nonlinearities) very similar to real system, but in cases of more complex nonlinearities, the model gives worse results and the simplification can not be applied.
The model can be also used to produce the nonlinearities in order to create audio-testing and to observe the impact of various nonlinearities from listener's point of view.

Acknowledgements

Research described in the paper was supervised by Doc. F. Kadlec, FEE CTU in Prague and supported by the grant of the Czech Science Foundation no. 102/05/2054 ”Qualitative Aspects of Audiovisual Information Processing in Multimedia Systems” and research project MSM6840770014 ”Research in the Area of Prospective Information and Communication Technologies”.

References

[1] SCHETZEN, M.: *The Volterra and Wiener theories of nonlinear systems*, New York, John Wiley & Sons, 1980

[2] GREEST, M.C., HAWKSFORD, M.O.: Distortion analysis of nonlinear system with memory using maximum-length sequences. *IEE Proc.-Circuits Devices Syst.* October 1995,vol.142, no. 5., p. 345-350.

About Author . . .

Antonín Novák, born on 15th July, 1981, is a post-graduate at the Czech Technical University in Prague, Faculty of Electrical Engineering and at Université du Maine, Le Mans as part of a PhD. Cotutelle programme. His research interests include analysis and measurement of non-linear systems and digital signal processing. He received an Ing. degree in engineering from the Czech Technical University in 2006 at the departement of Radioelectronics and, in the same year, an equivalence of Master 2 Recherche from Ecole Doctorale de l'Univeristé du Maine.